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Algebraic systems, which are not closed with respect to their operations.

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Algebraic systems, which are not closed with respect to their operations.

Usually in algebra systems are considered which are closed with respect to their defining operations. For the solution of some problems in algebra it may be useful to consider systems for which this is not the case. Especially embedding problems may be treated in this way. We take as an example the embedding of a semigroup in a group. If we take multiplication and forming of the inverse as basic operations of a group, and $(xy)z = x(yz)$, $xx^{-1} = yy^{-1}$, $x(xx^{-1}) = x$ as axioms, we may consider a semigroup as a system with the same basic operations and satisfying the same axioms as a group, but not closed with respect to the basic operations. In this case the second basic operation is defined for no element at all. If we say that a system, which is not closed, satisfies an axiom, we mean that it needs only to satisfy it if the occurring operations are defined. So this embedding problem may be subsumed under a general embedding problem of algebraic systems into closed algebraic systems of the same type.

Following B.H.Neumann [2] we call algebraic systems which are (resp. are not) closed with respect to their operations <u>full algebras</u> (<u>resp. partial algebras</u>).

Still another feature of the use of partial algebras may be mentioned. It is often possible to prove general statements about algebraic structures provided they are defined by axiom systems of a relatively simple logical structure. So e.g. it may be that existential statements in the axioms are not allowed. Now it may be that a certain algebraic structure cannot be defined by such axioms if the algebras of the structures are considered as full algebras, but that it is possible to do so if they are considered as partial algebras. As an example we take fields. If we take addition, subtraction and multiplication as basic operations fields are full algebras, but the axioms expressing the existence of an operation of division involve existential statements and it is not difficult to prove that fields as full algebras cannot be defined by an axiom system of a certain sufficiently simple nature, not to be described at this moment. If we add the operation of division to the list of basic operations it is easy to determine an axiom system for fields without existential statements, but fields no longer are full algebras, division not being defined if the divisor is equal to zero.

We discuss a general embedding theorem, due to B.H.Neumann. To be able to formulate this theorem we state that by an algebra A we mean a set on which finitary operations are defined belonging to a set V of operations of an arbitrary cardinal number. The set A is called the carrier of the algebra A. The domain, on which such an n-ary operation is defined is a, possibly empty, subset of the set of all n-tuples of elements of A. If for all operations this domain is the whole set of all n-tuples of elements of A, we call the algebra a full algebra. If it is desirable to express that an algebra needs not to be a full algebra, we call it a partial algebra. If we wish to denote the operations, we call A a V-algebra. A subalgebra of A is a subset B of A with the same operations as A; the domain of an n-ary operation is some subset of the intersection of the domain of that operation in A and the set of all n-tuples of elements of B. We note that every subset of A may be the carrier of a subalgebra of A, e.g. with empty domain for all operations. To every operation of V we associate an operation symbol with the same number of variable-places as the number of variables in the corresponding operation of V. With these operation symbols and a set of symbols, called variables, we form V-polynomials of order n recursively by substituting polynomials of order $<n$, at least one of which is of order n-1, on the variable-places of an operation symbol. The variables are polynomials of order zero. We define substitution of elements and operations of a V-algebra for the variables and operation symbols occurring in a V-polynomial in the obvious way. By an axiom-system Q we mean a set of statements, called axioms, each of which is a statement of propositional calculus in which identities of polynomials are substituted for the propositional variables. An algebra A is said to satisfy an axiom if every substitution of elements of A for the variables occurring in the axiom and of the operations of A for the corresponding operation symbols yields a correct statement about elements of A. If A satisfies all axioms of an axiom system Q we call A a Q-V-algebra. For a more detailed exposition of the notions of this paragraph compare Peremans [3] ; in that paper only full algebras are discussed.

Concerning the subalgebras of an algebra A we remark that given a subset B of A there exists a maximal subalgebra of A with carrier B; i.e. the algebra with carrier B in which the domain of every(n-ary) operation is the set of all those n-tuples of elements of B, for which the operation is defined and has its image in B. If this domain is the intersection of the set of all n-tuples of B and the domain of the operation in A, the subalgebra is called closed in A. Clearly a closed subalgebra of a full algebra is itself a full algebra. We remark, that strictly speaking there is an ambiguity in calling B a subalgebra of A, for in general there are several subalgebras of A with the same carrier. Moreover "closed" is used here in a sense somewhat different from the sense in which it was used in the introduction of this paper. Now Neumann's theorem reads as follows.

The partial Q-V-algebra A is isomorphic to a subalgebra of a full Q-V- algebra if (and trivially only if) every finite subalgebra of A is isomorphic to a subalgebra of a full Q-V-algebra.

If A is isomorphic to a subalgebra of B we say that A is embeddable in B.

The proof of this theorem uses a technique analogous to that used for the construction of free algebras. We take a set of variables which stands in a one-to-one correspondence with the elements of A and form the set P of the polynomials in those variables. In P an equivalence relation has to be constructed. For a finite subset U of P we consider the variables, finite in number, which occur in the polynomials of P, and the finite subset E of A corresponding to those variables. Now the maximal subalgebra of A with carrier E is embeddable in a full Q-V-algebra E'. Such an embedding induces an equivalence in U, two polynomials of U being equivalent if and only if substitution of the corresponding elements of E for the variables occurring in the polynomials and of the operations of E' for the operation symbols yields the same element of E'. The main problem of the proof is the construction of an equivalence in P such that this equivalence induces in every finite subset U of P an equivalence of the type just described. It is a curious fact that if we formulate this problem in a suitable way we get a well-known topological theorem.

To every finite subset U of P we associate the set R(U) consisting of the equivalence relations on U induced by all the above-mentioned embeddings of E in full Q-V-algebras. The set R(U) is obviously finite, U being finite, and not empty, because there exists at least one embedding of E into a full Q-V-algebra. Furthermore if U and V are finite subsets of P and $U \subset V$, and if E and F are the subsets of A associated with U (resp. V), clearly $E \subset F$ and every embedding of F into a full Q-V-algebra F' induces an embedding of E in F'. Obviously the equivalence relation on U induced by the embedding of E in F' is the restriction to U of the equivalence relation on V induced by the embedding of F in F'. So we get a mapping of R(V) in R(U), which we denote by f_U^V. If $U \subset V \subset W$ are subsets of P, we have $f_U^W = f_U^V f_V^W$. These facts are closely related to the concept of an inverse mapping system in topology.

Let L be a directed set, i.e. a partially ordered set, such that to all elements $\mu, \nu \in L$ there exists at least one $\lambda \in L$ such that $\mu \leqq \lambda$ and $\nu \leqq \lambda$. Let $M = \{U_\nu\}$ be a class of topological spaces indexed by the directed set L. If to every pair λ, μ of L with $\lambda \geqq \mu$, there exists a continuous mapping f_μ^λ of U_λ into U_μ, such that for $\lambda \geqq \mu \geqq \nu$ we have $f_\nu^\lambda = f_\nu^\mu f_\mu^\lambda$, we call the class M together with the mappings f_μ^λ an inverse mapping system. By the limit space of an inverse mapping system we mean the subset of the Cartesian product $\prod_\nu U_\nu$ of all U_ν consisting of all points $x=(x_\nu)$ such that for every pair $\lambda, \mu \in L$ with $\lambda \geqq \mu$ we have $f_\mu^\lambda x_\lambda = x_\mu$. A theorem of Steenrod [4] asserts that if all U_ν are compact and not empty, the limit space is not empty.

To apply the theorem of Steenrod to our problem we take for L the set of all finite subsets of P partially ordered by inclusion. Obviously L is a directed set, it being a lattice. For M we take the class $\{R(U)\}$ indexed by U, and for the mappings we take the mappings f_U^V defined above. This gives us an inverse mapping system, if we topologize the finite non-empty sets R(U) by the discrete topology. The sets R(U) then are compact and not empty and we can apply the theorem of Steenrod. This means that we can choose in R(U) for every finite subset U of P an element r_U (i.e. an equivalence relation in U of the type described above) such that $U \subset V$ implies that r_U is the restriction to U of r_V. We now define an equivalence relation r on P by stating that two elements p and q of P are equivalent if and only if in the set W consisting of the two elements p and q the elements p and q are equivalent in r_W. It is not difficult to show that r is an equivalence relation on P and that for every finite subset U of P we have that r_U is the restriction of r to U.

We now give a proof of the theorem of Steenrod for the case that all U_ν are finite. In this case it is not a topological theorem, but the proof makes use of concepts used in topology (see Lefschetz [1]).

We call R the Cartesian product of all U_ν and we form for all $\lambda, \mu \in L$ with $\lambda \geqq \mu$ the sets S_μ^λ consitsting of those elements $x = (x_\nu) \in R$ for which $f_\mu^\lambda x_\lambda = x_\mu$. We first prove that the intersection of a finite number of sets S_μ^λ is not empty. Let these sets be $S_{\mu_1}^{\lambda_1}, \dots, S_{\mu_n}^{\lambda_n}$ with $\lambda_i \geqq \mu_i (i = 1, \dots, n)$. Because L is a directed set we may choose a $\lambda_0 \in L$ such that $\lambda_0 \geqq \lambda_i$ for $i = 1, \dots, n$ and a point $z_{\lambda_0} \in U_{\lambda_0}$. We define $z_{\lambda_i} = f_{\lambda_i}^{\lambda_0} z_{\lambda_0}$ and $z_{\mu_i} = f_{\mu_i}^{\lambda_0} z_{\lambda_0}$

for i=1,...n. We then have $f_{\mu_i}^{\lambda_i} \; z_{\lambda_i} = f_{\mu_i}^{\lambda_i} \; f_{\lambda_i}^{\lambda_o} \; z_{\lambda_o} = f_{\mu_i}^{\lambda_o} \; z_{\lambda_o} = z_{\mu_i}$ for i = 1,..., n. Every point $x = (x_\nu)$ with $x_{\lambda_i} = z_{\lambda_i}$ and $x_{\mu_i} = z_{\mu_i}$ (i=1,..., n) is an element of the intersection of $S_{\mu_1}^{\lambda_1}, \ldots, S_{\mu_n}^{\lambda_n}$, which proves our assertion.

We call S the collection of all S_μ^λ and H the family of all those collections C of subsets of R such that $S \subset C$ and such that every finite subcollection of C has a non-empty intersection. We have proved $S \in H$.

If $A_1 A_2, \ldots$ is an infinite sequence of elements of H and if $A_n \subset A_{n+1}$ for all n, we have $A = \bigcup_{n=1}^{\infty} A_n \in H$, because a finite subcollection of A is a subcollection of one of the A_n and therefore has a non-empty intersection. Furthermore $S \subset A$.

We now can apply Zorn's theorem, which asserts that there exists a maximal element $G \in H$. For G we have:

1°. The intersection I of a finite subcollection of G is an element of G.

If this is not the case, we take the collection G_1 consisting of the elements of G and of I. If a finite subcollection of G_1 does not contain I it is a subcollection of G and therefore has a non-empty intersection. If a finite subcollection C of G_1 contains I, its intersection is the same as the intersection of the given finite number of elements of G with intersection I and the elements of C different from I, which are elements of G too. Therefore C has a non-empty intersection. Furthermore $S \subset G_1$, so $G_1 \in H$, which contradicts the maximality of G.

2°. A subset T of R which has a non-empty intersection with every element of G is itself an element of G.

If this is not the case, we take the collection G_2 consisting of the elements of G and of T. If a finite subcollection of G_2 does not contain T it is a subcollection of G and therefore has a non-empty intersection. If a finite subcollection C of G_2 contains T, its intersection is the same as the intersection of T and the intersection of the elements of C different from T, the latter intersection being an element of G by 1°. This intersection is not empty by assumption. Furthermore $S \subset G_2$, so $G_2 \in H$, which contradicts the maximality of G.

We now fix an element $\lambda \in L$. For every element of G we consider its projection in U_λ. The number of subsets of U_λ is finite, U_λ being finite; thus also the number of subsets of U_λ which are projections of elements of G is finite. For every such subset we choose an element of G having this subset as projection; this yields a finite number of elements of G, which therefore have a non-empty intersection. Their projections also have a non-empty intersection. So there exists an element $y_\lambda \in U_\lambda$, which is an element of the projection of every element of G. We now take $y = (y_\nu) \in R$ and assert that this point satisfies the requirements of Steenrod's theorem.

To prove this, we take for a fixed $\lambda \in R$ the set Q_λ of those points $x = (x_\nu) \in R$ for which $x_\lambda = y_\lambda$. Now Q_λ has a non-empty intersection with every element of G and so by 2^0 is an element of G. For $\lambda, \mu \in R$ with $\lambda \geqq \mu$ we consider $Q_\lambda \cap Q_\mu \cap S_{\lambda\mu}$. This set is not empty; we take an element $x = (x_\nu)$ out of it. We then have $x_\lambda = y_\lambda$, $x_\mu = y_\mu$ and $f^\lambda_\mu \; x_\lambda = x_\mu$, so $f^\lambda_\mu \; y_\lambda = y_\mu$ and Steenrod's theorem is proved.

To construct the full Q-V-algebra, in which A may be embedded, we take the set B, whose elements are the equivalence classes of P under the equivalence r. To make B a V-algebra, we proceed as follows. If $\alpha_1, \dots, \alpha_n$ are elements of B and O an n-ary operator of V, we take representants $a_1, \dots, a_n$ out of the classes $\alpha_1, \dots, \alpha_n$ and the operation symbol O_1 corresponding to the operator O. Now we define $O(\alpha_1, \dots, \alpha_n)$ to be the class containing the polynomial $O_1(a_1, \dots, a_n)$. To justify this definition we have to prove that it is independent of the choice of the representants out of $\alpha_1, \dots, \alpha_n$. This amounts to the theorem that if $a_i \sim b_i$ (i=1,...,n; the symbol $\sim$ means "equivalent with respect to r"), then $O_1(a_1, \dots, a_n) \sim O_1(b_1, \dots b_n)$. This is easily proved by restricting r to a finite subset of P, containing $a_1, \dots, a_n$, $b_1, \dots, b_n$, $O_1(a_1, \dots, a_n)$ and $O_1(b_1, \dots, b_n)$ and interpreting this equivalence with use of an embedding of a finite subalgebra of A in a full Q-V-algebra. In an analogous way we prove that B, which obviously is a full algebra, is a Q-V-algebra (in an axiom only a finite number of variables occur). Finally to prove that A is isomorphic with a subalgebra of B, we associate with an element a of A that class of B that contains the variable (polynomial of order zero) corresponding to a. It is not difficult to prove that this correspondence is one-to-one and that it is an isomorphism. Here the fact is used that the embeddings of finite subalgebras are effected on the maximal subalgebras with a given carrier.

There are numerous applications of the embedding theorem. We mention one of the applications given by Neumann. The relation of total order may be defined by an operator which associates with two elements their maximum. It is easy to see that the concept of a totally ordered group satisfies the requirements of a Q-V-algebra. Now by Neumann's theorem we infer:

A group can be totally ordered if and only if every finitely generated subgroup can be totally ordered.

By using some tricks it is possible to apply the theorem to cases in which the algebras concerned are at first sight no Q-V-algebras. So by introducing a dumming element ∞ we can make division rings full Q-V-algebras and we get that a ring can be embedded in a division ring if every finitely generated subring can be embedded in a division ring.

An important embedding problem in the literature is that for cardinal algebras (see Tarski [5]). Unfortunately I do not see how to treat this problem in such a way that Neumann's theorem can be applied. A cardinal algebra is a full algebra with two operations, one binary (written +) and one denumerably infinitary (written Σ) satisfying a set of axioms, given in Tarski [5]. A generalized cardinal algebra is a corresponding partial algebra satisfying some weak closure postulates. Now Tarski proves that every generalized cardinal algebra can be embedded in a cardinal algebra. His proof gives a construction of that cardinal algebra and is adapted to the special properties of this type of algebra; moreover he proves more, namely that the embedding may be effected in such a way, that every element a of the full algebra may be written as $a = \Sigma a_i$ with the a_i in the partial algebra. The pricipal reason, why Neumann's theorem is not applicable is the occurrence of an infinitary operation; the proof of his theorem makes an essential use of the fact that all operations are finitary (the "topological" lemma!). Now it is possible to express the operation Σ in the operation +, as Tarski shows, but this procedure involves such an essential complication in the axiom system (existence of infinite sequences of elements), that if we restrict ourselves to the operation +, the axiom system is such that Neumann's theorem cannot be applied. It may be hoped that it will appear possible to extend Neumann's theorem such that infinitary operations are allowed and such that it covers the case of cardinal algebras.

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